RECURSIVE SEQUENCES

LAST NAME	FIRST NAME	DATE

1 (5 points). A sequence is defined by

$$\begin{aligned} a_1 &= 2 \\ a_n &= a_{n-1}^2 - 3a_{n-1} \text{ if } n > 1 \end{aligned}$$

Find second, third, fourth, fifth, and sixth members of the sequence.

$a_2 =$

$a_3 =$

$a_4 =$

$a_5 =$

$a_6 =$

2 (5 points). A sequence is defined by

$$
\begin{aligned}
b_1 &= 3 \\
b_2 &= 5 \\
b_n &= (b_{n-1} - b_{n-2})^2 \text{ if } n > 2
\end{aligned}
$$

Find the initial segment of this sequence:

$b_3 =$

$b_4 =$

$b_5 =$

$b_6 =$

$b_7 =$

3. The infamous Collatz conjecture states that the recursive sequence C_n always reaches 1 (or, alternatively, the 4-2-1 cycle), regardless of which positive integer it starts from. The sequence is defined by

$$C_1 = k, \text{ a positive integer}$$

$$C_n = \begin{cases} C_{n-1}/2 & \text{if } C_{n-1} \text{ is even} \\ 3C_{n-1}+1 & \text{if } C_{n-1} \text{ is odd} \end{cases} \quad \text{if } n > 1$$

(a) Write out the elements of the Collatz sequence until you reach 1, given that $C_1 = 12$

(b) Do it again, now for $C_1 = 19$

4 (4 points). Given two real numbers numbers s and r, a sequence is defined by

$$\begin{aligned} x_1 &= r \\ x_2 &= s \\ x_n &= 2x_{n-1} - x_{n-2} \text{ if } n > 2 \end{aligned}$$

Find the third, fourth, fifth, and sixth members of the sequence. Simplify each answer by removing parentheses and combining like terms.

$x_3 =$

$x_4 =$

$x_5 =$

$x_6 =$

In 1946, a Hungarian and American mathematician John von Neumann programmed what was perhaps the first pseudo-random number generator, now known as the *middle-square method*. A variation of it can be described like so: take a four digit number, square it, and then select the middle four digits, padding the result with a zero on the left if needed. This creates a recursive sequence V_n, where V_1 is the initial value, known as *the seed*, and V_n are the middle four digits of V_{n-1}^2.

For example, if $V_1 = 1234$:

$V_1^2 = 01\mathbf{5227}56$ and $V_2 = 5227$,

$V_2^2 = 27\mathbf{3215}29$ and $V_3 = 3215$,

$V_3^2 = 10\mathbf{3362}25$ and $V_3 = 3362$, and so on.

5. If $V_1 = 3066$, find the first 10 elements of the sequence V_n.

How would you describe the long-term behavior of this sequence?

Find V_{1001}

6. If V_n is a von Neumann middle-square sequence and $V_1 = 7492$, find the first 10 elements of the sequence V_n.

How would you describe the long-term behavior of this sequence?

Find V_{1001}

7. Let the recursive sequence M_n be defined as follows:

$M_1 = 0$

$M_n = M_{n-1}^2 + C$, where C is some constant.

Find M_9, M_{10}, and M_{11} if $C = 0.1$

Make a conjecture about the approximate value of M_{1001}.

Pell numbers are an infinite sequence of integers, known since ancient times. They are similar to Fibonacci numbers, but have their unique claim to fame in that they comprise the denominators of the closest rational approximations to the square root of 2:

$$\frac{1}{1} \quad \frac{3}{2} \quad \frac{7}{5} \quad \frac{17}{12} \quad \frac{41}{29} \quad \cdots$$

8. Define the sequence of Pell numbers with the following recursive relation:

$P_0 = 0$

$P_1 = 1$

$P_n = 2P_{n-1} + P_{n-2}$ for $n > 1$

Find P_{17}

Just like the Fibonacci sequence, this one has a closed form formula. Find

$$\frac{(1+\sqrt{2})^{17} - (1-\sqrt{2})^{17}}{2\sqrt{2}}$$

and compare it with your answer for P_{17}.

9. Let the sequence D_n be defined as follows:

$D_1 = 1$

$D_n =$ the sum of all the previous elements of the sequence.

Find D_2 through D_{10}.

Describe an easier way to find D_n than to follow the recursive formula. Use it to compute D_{32}.

10. Recall that the least common multiple (LCM) of two positive integers x and y is the smallest integer m which is divisible by x as well as by y. For example,
the LCM of 5 and 6 is 30,
the LCM of 8 and 6 is 24,
the LCM of 49 and 7 is 49.

Define the recursive sequence L_n as follows:

$L_1 = 1$

$L_n =$ LCM of n and L_{n-1} for $n > 1$.

Find L_2 through L_{10}.